A few derived laws:

Lemma 1:

i) $-(-a) = a$

$(a^{-1})^{-1} = a$ if $a \neq 0$

ii) $(-a) + (-b) = -(a+b)$

$a^{-1} \cdot b^{-1} = (a \cdot b)^{-1}$ if $a, b \neq 0$

iii) $a \cdot 0 = 0$

$a \cdot (-b) = -a \cdot b$

$(-a) \cdot (-b) = a \cdot b$

$a \cdot b = 0 \iff (a = 0 \text{ or } b = 0)$

Proof of Lemma 1:

i) Claim

$(a^{-1})^{-1} = a$ for $a \neq 0$

Indeed

$1 = a \cdot a^{-1}$ inv. elem.

$1 = a^{-1} \cdot a$ comm.

$\rightarrow$ a is inverse element with respect to a^{-1}

ii) Claim: $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$ for $a, b \neq 0$

$\rightarrow$ have to show: $(a \cdot b) \cdot (a^{-1} \cdot b^{-1}) = 1$

Indeed

$$(a\cdot b)\cdot(a^{-1}\cdot b^{-1}) = (b\cdot a)\cdot(a^{-1}\cdot b^{-1})$$
$$= b\cdot((a\cdot a^{-1})\cdot b^{-1})$$
$$= b\cdot(1\cdot b^{-1}) = b\cdot b^{-1} = 1$$

iii) Claim : $a\cdot 0 = 0$

Indeed

$$(a\cdot 0) = a\cdot(0+0)$$
$$= a\cdot 0 + a\cdot 0$$

$$\rightarrow (a\cdot 0)-(a\cdot 0) = (a\cdot 0 + a\cdot 0) - (a\cdot 0)$$
$$= a\cdot 0 + (a\cdot 0 - a\cdot 0)$$

Using inverse element with respect to $+$ we get

$$0 = a\cdot 0 + 0$$
$$= a\cdot 0 \qquad \text{neutr. } +$$

Claim: $a\cdot(-b) = -a\cdot b$

This holds as

$$a\cdot b + a\cdot(-b) = a\cdot(b-b) \qquad \text{Distr.}$$
$$= a\cdot 0 \qquad \text{inv. } +$$
$$= 0 \qquad \text{(see above)}$$

Claim: $a\cdot b = 0 \Rightarrow (a=0 \text{ or } b=0)$

suppose $a\cdot b = 0$

Case 1: $b=0 \rightarrow$ we are done

Case 2: $b \neq 0$, then we have to prove $a=0$

$$a \cdot b = 0 \Rightarrow (a \cdot b) \cdot b^{-1} = 0 \cdot b^{-1} \quad \text{as } b \neq 0$$

$$a \cdot (b \cdot b^{-1}) = 0 \quad \text{(see above)}$$

$$a \cdot 1 = 0$$

$$a = 0$$

$\square$

B) Ordering axioms

On $\mathbb{R}$ there exists a relation $<$.

For pairs (a, b): $a<b$ satisfying following axioms:

trichotomy	either $a<b$ or $a=b$ or $b<a$
transitive	$(a<b$ and $b<c) \Rightarrow a<c$
compatible with $+$	$a<b \Rightarrow a+c<b+c$
compatible with $\cdot$	$(a<b$ and $0<c) \Rightarrow a \cdot c < b \cdot c$

Notation: $a<b$ and $b>a$ are equivalent

$(a<b$ or $a=b)$ is equivalent to $a \leq b$

$(a>b$ or $a=b)$ is equivalent to $a \geq b$

Lemma 2:

i) $(a<0$ and $b<0) \Rightarrow a+b<0$

$(a>0$ and $b>0) \Rightarrow a+b>0$

$a<0 \Leftrightarrow -a>0$

ii) $a \cdot b > 0 \iff ((a > 0 \text{ and } b > 0)$
$\text{or } (a < 0 \text{ and } b < 0))$

$a \cdot b < 0 \iff ((a > 0 \text{ and } b < 0) \text{ or }$
$(a < 0 \text{ and } b > 0))$

iii) $0 < 1$

$a < 0 \iff a^{-1} < 0$

<u>Proof of Lemma 2:</u>

i) Claim: $(a < 0 \text{ and } b < 0) \Rightarrow a + b < 0$

$$\left.\begin{array}{l} a < 0 \Rightarrow a + b < b \\ \quad \text{compatible with } + \\ b < 0 \end{array}\right\} \begin{array}{l} a + b < 0 \\ \text{transitive} \end{array}$$

Claim: $a < 0 \Rightarrow -a > 0$

$a < 0 \Rightarrow \underbrace{a + (-a)}_{= 0 \quad \text{inv. } +} < -a$ Compatible with $+$

$\rightarrow -a > 0$

ii) Lemma 1 iii)

$(a = 0 \text{ or } b = 0) \iff a \cdot b = 0$ (1)

Trichotomy $\rightarrow$ need to prove

$(a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$

$\Rightarrow a \cdot b > 0$ (2)

as well as

$(a > 0 \text{ and } b < 0) \text{ or } (a < 0 \text{ and } b > 0)$

$\Rightarrow a \cdot b < 0$ (3)

Remark: two strategies for proving $A \Leftrightarrow B$

1. prove $A \Rightarrow B$
   prove $B \Rightarrow A$

2. prove $A \Rightarrow B$
   prove $\neg A \Rightarrow \neg B$

Proving (1), (2) and (3) will imply inverse of (2):

$$\neg((a>0 \text{ and } b>0) \text{ or } (a<0 \text{ and } b<0))$$
$$\Rightarrow \neg(a \cdot b > 0)$$

and similarly the inverse of (3)

$\rightarrow$ it suffices to prove statements (2) and (3)

We prove here (3):

due to symmetry, it is enough to prove

$(a>0 \text{ and } b<0) \Rightarrow a \cdot b < 0$

Indeed $(a>0 \text{ and } b<0)$

$\Rightarrow (a>0 \text{ and } -b>0)$ due to i)

$\Rightarrow a \cdot (-b) > 0 \cdot (-b)$ (*) compatible with $\cdot$

Lemma 1 $\rightarrow$ $a \cdot (-b) = -ab$
$0 \cdot (-b) = 0$

$\Rightarrow$ (*) $\Leftrightarrow -ab > 0 \Leftrightarrow ab < 0$ due to i)

iii) Claim: $0<1$

field axioms give $0 \neq 1$

Due to trichotomy it suffices to prove that $\neg(1<0)$.

Suppose $1<0 \overset{ii)}{\Longrightarrow} \underbrace{1 \cdot 1}_{=1} > 0$ (1 is neutr. $\cdot$)

$\longrightarrow$ contradiction due to trichotomy

$\Longrightarrow$ $1>0$ (and therefore also $0<1<2<\dots$ by comp. $+$) $\square$

Review so far:

3 classes of axioms

A) The field axioms ✓

B) Ordering axioms ✓

$\rightarrow$ C) Completeness axiom

$\mathbb{Q}$ and $\mathbb{R}$ are both fields. The difference is that $\mathbb{R}$ is "complete" with respect to ordering:

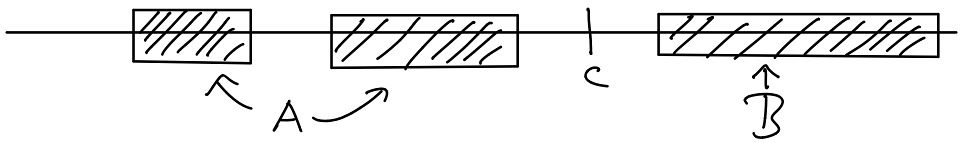
Let $A, B \subset \mathbb{R}$ be non-empty sets

Then

$$\forall\, a \in A,\ b \in B:\ a \leq b$$

there is a number $c \in \mathbb{R}$ with

$$\forall\, a \in A,\ b \in B:\ a \leq c \leq b$$



<u>Note:</u> $\mathbb{Q}$ is not complete with respect to ordering

<u>Proposition 1.1:</u>

i) given $a \in \mathbb{R}$, $a > 0$: $\exists\, n \in \mathbb{N}$ with $n > a$

ii) for $x, y \in \mathbb{R}$ with $x < y$, we have $\exists\, z \in \mathbb{Q}$ such that $x < z < y$

iii) $\forall\, x, y \geq 0$: $x \leq y \iff x^2 \leq y^2$

iv) $\exists\, c \in \mathbb{R}$: $c^2 = 2$, $c > 0$.

<u>Proof:</u>

i) Lemma 2 iii) $\Rightarrow \mathbb{N} \subset \mathbb{R}$

Set $A = \{\alpha \in \mathbb{R} \mid \alpha \leq a,\ \alpha > n\ \forall\, n \in \mathbb{N}\}$

Then $\mathbb{N}$ and A are non-empty sets such that

$\forall\, n \in \mathbb{N}$ and $\alpha \in A$: $n \leq \alpha$

Completenes axiom $\Rightarrow \exists\, c \in \mathbb{R}$: $n \leq c \leq \alpha$ $\forall\, n \in \mathbb{N},\ \alpha \in A$

Then $\exists\, m \in \mathbb{N}$: $m > c-1$ (otherwise $c-1 \in A$)

But then $m+1 > c$ $\lightning$ $\Rightarrow A$ is empty

ii) i), compatibility with $\cdot$ gives

$n^{-1} \cdot n > n^{-1} \cdot a \Rightarrow a^{-1} > n^{-1}$

set $a = (y-x)^{-1}$, Lemma 2 iii) $\Rightarrow a > 0$

and hence $\frac{1}{n} < y - x$

choose $m \in \mathbb{N}$ such that $\frac{m}{n} > x$ and $\frac{(m-1)}{n} < x$

Then $\frac{m}{n} < y$

iii) exercise

iv) Lemma 2 iii) $\Rightarrow \mathbb{Q} \subset \mathbb{R}$

set $A = \{a \in \mathbb{Q} \mid 1 \leq a \leq a^2 < 2\}$,

$B = \{b \in \mathbb{Q} \mid 1 \leq b \leq 2,\ b^2 \geq 2\}$

$\Rightarrow 1 \in A,\ 2 \in B \Rightarrow A \neq \emptyset \neq B$

iii) $\Rightarrow \forall a \in A,\ b \in B : a < b$

Completeness axiom gives $c \in \mathbb{R}$ with the property: $\forall a \in A,\ b \in B : a \leq c \leq b$ $(*)$

$\Rightarrow 1 \leq c \leq 2$

"c is unique":

suppose $\exists\ c_1 < c_2$ in $\mathbb{R}$ satisfying $(*)$

$\rightarrow \exists\ c_1, c_2 \in \mathbb{Q}$ satisfying $(*)$

(between any two real numbers there is a rational number)

Then $c_0 = \frac{c_1 + c_2}{2} \in \mathbb{Q}$ and

$\forall\ a \in A,\ b \in B:\ a \leq c_1 < c_0 = \frac{c_1 + c_0}{2} < c_2 \leq b$

$c_0 \in A \cup B$. If $c_0 \in A$ (*) cannot hold for c_1,
if $c_0 \in B$ (*) cannot hold for c_2 $\lightning$

"$c^2 = 2$":

$\forall\ a \in A,\ b \in B$ we have

a) $2 - c^2 \leq b^2 - c^2 \leq b^2 - a^2 = (b-a)\underbrace{(b+a)}_{\leq 4} \leq 4\underbrace{(b-a)}_{>0}$

b) $2 - c^2 \geq a^2 - c^2 \geq a^2 - b^2 = \underbrace{(a-b)}_{<0}\underbrace{(b+a)}_{\leq 4}$

a) $\Rightarrow c^2 \geq 2$

Why? Suppose the opposite is true: $c^2 < 2$

set $\varepsilon = 2 - c^2 > 0$,

then by ii) $\exists\ a, b \in \mathbb{Q}$ with $c < b < c + \frac{\varepsilon}{8}$,

$c - \frac{\varepsilon}{8} < a < c$

$\Rightarrow 4(b-a) < 4\left(c + \frac{\varepsilon}{8} - c + \frac{\varepsilon}{8}\right) = \varepsilon$

but $4(b-a) \geq 2 - c^2 = \varepsilon$ $\lightning$

similarly b) $\Rightarrow c^2 \leq 2$

Altogether, we then get: $c^2 = 2$

$\square$